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Random walks on three-strand braids and on related hyperbolic groups

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Abstract
We investigate the statistical properties of random walks on the simplest nontrivial braid group B_3, and on related hyperbolic groups. We provide a method using Cayley graphs of groups allowing us to compute explicitly the probability distribution of the basic statistical characteristics of random trajectories—the drift and the return probability. The action of the groups under consideration in the hyperbolic plane is investigated, and the distribution of a geometric invariant—the hyperbolic distance—is analysed. It is shown that a random walk on B_3 can be viewed as a 'magnetic random walk' on the group $PSL(2, \mathbb{Z})$.

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1. Introduction

The investigation of the statistical aspects of braids and, in general, of homotopy groups seems to be a relatively new issue in condensed matter physics. The main difficulties in the statistical study of topology of linear uncrossable objects are due to two facts: (a) the topological constraints are non-local and (b) different entanglements do not commute. The difficulty (a) has been resolved so far by introducing Abelian Gauss-like topological invariants, counting windings of one chain around the other. This Abelian approximation loses the very rich content of property (b).

Before entering the details of the problems under consideration, it makes sense to pay special attention to a few physical examples where the non-Abelian structure of the phase spaces of uncrossable linear objects with topological constraints plays a crucial role in the observable properties of these physical systems.

The most elaborated examples belong to the area of statistics of polymer systems with topological constraints (see, for example, [1] and references therein). The chain-like structure of macromolecules causes the so-called linear memory (i.e. fixed position of each monomer unit along the chain), which, besides the standard properties (such as the low translational

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entropy and large space fluctuations) leads to the fact that different parts of polymer molecules fluctuating in space cannot pass through one another without chain rupture. For the system of non-phantom closed chains this means that only those chain conformations are available which can be transformed into one another continuously. The influence of topological constraints on the physical properties of polymer networks is manifested, for example, in the strong deviation of the stress–strain dependence from a classical elasticity theory in polymer rubbers [2]. The modern theories [3] taking properly into account the noncommutative character of multiple entanglements in polymer networks describe the observed experimental data with very good precision. Another example in this field deals with the influence of topological constraints on the fractal structure of strongly collapsed unknotted ring polymer. In the set of works [4] it has been shown that the *global* topological condition of the absence of knots on a collapsed polymer chain has a strong influence on all *local* scales making each sub-part of the polymer ring almost unentangled. The investigation of the corresponding structure, called a 'crumpled globule', is impossible in the framework of the Abelian description of topological constraints. The principal results both for the problems of high elasticity of polymer rubbers and for the statistics of collapsed polymer rings have been achieved in the framework of the model 'random walk in an array of obstacles' [5]) which is nothing other than a physical interpretation of the random walk on the *free group*.

Another very transparent example of manifestations of topological constraints in condensed matter physics deals with the dynamical properties of vortex glasses in high temperature superconductors [6]. In CuO_2-based high-T_c superconductors in fields less than H_{c2} there exists a region where the Abrikosov flux lattice is molten, but the sample of the superconductor demonstrates the absence of conductivity. This effect has been explained by the highly entangled state of the *braid* of flux lines due to their topological constraints [7]. The recent contribution to this subject [8] develops a symbolic language which permits construction of objects with a braid-like topology in 2 + 1 dimensions and to solve the simplest statistical problems where the noncommutative character of topological constraints is explicitly taken into account.

The aim of this work is to develop constructive methods of investigation of the statistical properties of random processes on the simplest nontrivial noncommutative braid and braid-like groups, emerging in various problems of condensed matter physics where the non-Abelian character of the phase space plays a crucial role [9]. To be more specific, the paper is devoted to a study of random walks on the modular group $PSL(2, \mathbb{Z})$ and some closely related groups: the simplest nontrivial three-strand braid group B_3, the Hecke groups H_q and the free groups F_n (all definitions are given below). We examine simultaneously the limiting distribution of random walks on Cayley graphs of these groups as well as on the embedding of these groups in the hyperbolic plane. We analyse the statistical properties of random walks on the Cayley graphs of the above-mentioned groups both in a metric of words and in the natural metric in the hyperbolic plane.

To establish links of the particular questions addressed in the current work with real physical problems we discuss in the conclusion the physical significance of the results obtained in our contribution. Let us mention briefly that the two most important physical questions in statistical topology are: (a) the evaluation of the knot (or braid) entropy (i.e. the volume of available configurational space for fixed topological state of the knot (or the braid)), and (b) the computation of expectation values of topological invariants.

The statistics of Markov chains on the group $PSL(2, \mathbb{R})$ and its subgroups has been extensively studied in the mathematical literature. Among the known results linked to the theme of our work we can mention: (a) the central limit theorem for Markov multiplicative processes on discrete subgroups of the group $PSL(2, \mathbb{R})$ [10, 11], (b) investigation of the boundary of some noncommutative groups [12, 13], as well as determination of various numerical characteristics of these groups [14, 15], (c) particular examples of the exact results for limiting distribution functions of random walks on Cayley graphs of free and modular groups [5, 16–18] and (d) conjectures concerning the return probability and drift on the braid groups B_3 [19] and B_n [20].

In this paper, we compute the *drift* and the *return probability* for symmetric random walks (in a metric of words) on the groups H_q and B_3. These two characteristics (the drift and the return probability) have natural topological interpretations: the drift is mainly related to the mean value of the topological invariant while the logarithm of the return probability gives the entropy of the trivial topological state (see also the discussion in the conclusion).

Moreover, as has been said, we pay special attention to the statistics of random walks on the embeddings of the groups $PSL(2, \mathbb{Z})$, B_3, H_q, F_n in the hyperbolic plane. To be specific, we study a 2×2 matrix representation of these groups and consider their homographic action[3] on the hyperbolic plane $\mathcal{H}$. This allows us to embed the Cayley graphs in $\mathcal{H}$ and to define isometric hyperbolic lattices. Taking advantage of the hyperbolic metric on $\mathcal{H}$, we investigate the probability distribution of the geodesic distance between the ends of random processes with symmetric transition probabilities on these lattices embedded in $\mathcal{H}$. The problem under consideration is reduced to the study of the absolute value of a product of random matrices. This part of our investigation is semi-analytic and is based on the numerical results on the structure of the invariant distribution of geodesics at the boundary of $\mathcal{H}$. We found that the drift on a Cayley graph in a metric of words coincides after proper normalization with the drift on the corresponding isometric lattice of $\mathcal{H}$ in the natural hyperbolic metric. This result establishes a relation between two group invariants: (i) the irreducible length of a group element, which does not depend on the representation, and (ii) the hyperbolic distance associated with the same group element (directly linked to its absolute value), defined only for the matrix representation.

As an application of our results, we consider the relation between the distribution of Alexander knot invariants [21, 22] and the asymptotic behaviour of random walks over the elements of the simplest nontrivial braid group B_3. This class of problems arises naturally even beyond the aims of our particular investigation: the limiting behaviour of Markov chains on braid and so-called local groups can be regarded as the first step in a consistent development of harmonic analysis on multiconnected manifolds (Teichmüller spaces are an example).

The paper is structured as follows. In section 2, we give the basic definitions, introducing the different groups and their Cayley graphs. A general solution to the diffusion problem on these graphs, as well as exact computations of the drift and the return probability for B_3, are developed in section 3. Section 4 is devoted to the study of the action of these groups in the hyperbolic plane. A discussion of our results, the relation between the different approaches as well as some physical consequences are presented in section 5.

2. The groups $PSL(2, \mathbb{Z}), H_q, B_3, F_n$ and their Cayley graphs

2.1. Basic definitions

We consider a special class of hyperbolic-like groups which is the modular group $PSL(2, \mathbb{Z})$. Some of its generalizations are the Hecke groups H_q, and its central extension, the braid group B_3. By hyperbolic-like groups we mean a class broader than Gromov's hyperbolic groups [24], which prohibits central extensions like B_3. The important feature for us is just the exponential

[3] Due to the fact that the group B_3 is not a hyperbolic group and since the two-representation of B_3 is not unimodular, its homographic action is not faithful.

growth. We shall also recall the well-known properties of the free groups F_n (we denote by F_n the free product of n copies of $\mathbb{Z}_2$).

1. The modular group $PSL(2,\mathbb{Z})$ is a free product $\mathbb{Z}_2 \star \mathbb{Z}_3$ of the two cyclic groups of second (generated by a_2) and third (generated by b_3) orders. In a standard framing using generators S (inversion) and T (translation), the group $PSL(2,\mathbb{Z})$ is defined by the following relations:

$$(ST)^3 = b_3^3 = 1 \qquad S^2 = a_2^2 = 1. \tag{1}$$

The generators T and S of the modular group $PSL(2,\mathbb{Z})$ have a natural representation by unimodular matrices $\hat{T}$ and $\hat{S}$:

$$\hat{T} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad \hat{S} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{2}$$

2. In addition to the modular group $PSL(2,\mathbb{Z})$ we shall consider the so-called Hecke group H_q which 'interpolates' between the modular group $PSL(2,\mathbb{Z})$ (for $q = 3$) and the free group F_3 with three generators, the so-called Λ group (for $q = \infty$). The Hecke group H_q is isomorphic to $\mathbb{Z}_2 \star \mathbb{Z}_q$ (we denote by a_2 and b_q the generators of orders 2 and q). The Hecke group is defined by a straightforward generalization of relations (1)

$$(ST_q)^q = b_q^q = 1 \qquad S^2 = a_2^2 = 1 \tag{3}$$

and the generators T_q and S have the following matrix representation (compare to (2)):

$$\hat{T}_q = \begin{pmatrix} 1 & 2\cos\frac{\pi}{q} \\ 0 & 1 \end{pmatrix} \qquad \hat{S} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{4}$$

The parameter q takes the discrete values $q = 3, 4, 5, 6, \ldots$.

3. The braid group B_3 is defined by the commutation relations among generators $\{\sigma_1, \sigma_2\}$:

$$\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 \qquad \sigma_1\sigma_1^{-1} = \sigma_2\sigma_2^{-1} = \mathrm{e}. \tag{5}$$

In our further construction we shall repeatedly use another framing:

$$\tilde{a} = \sigma_1\sigma_2\sigma_1 \qquad \tilde{b} = \sigma_1\sigma_2. \tag{6}$$

The generators of the group B_3 can be represented by $PGL(2,\mathbb{R})$ matrices. To be more specific, the generators σ_1 and σ_2 in the Burau representation [21] read

$$\hat{\sigma}_1 = \begin{pmatrix} -t & 1 \\ 0 & 1 \end{pmatrix} \qquad \hat{\sigma}_2 = \begin{pmatrix} 1 & 0 \\ t & -t \end{pmatrix} \tag{7}$$

where t is the free parameter. It is more convenient to introduce the parameter u as follows:

$$u = \begin{cases} \sqrt{t} & \text{for} \quad t \geqslant 0 \\ \sqrt{-t} & \text{for} \quad t < 0 \end{cases}$$

and consider normalized generators ${}_u\hat{\sigma}_1$ and ${}_u\hat{\sigma}_2$ with determinant 1:

$${}_u\hat{\sigma}_1 = \begin{pmatrix} u & 1/u \\ 0 & 1/u \end{pmatrix} \qquad {}_u\hat{\sigma}_2 = \begin{pmatrix} 1/u & 0 \\ -u & u \end{pmatrix}. \tag{8}$$

The group generated by ${}_u\hat{\sigma}_1$ and ${}_u\hat{\sigma}_2$ will be denoted later on as $PSL(2,\mathbb{Z})_u$. Indeed it is just a deformation of $PSL(2,\mathbb{Z})$, which preserves all commutation relations. For $t = -1$ (i.e. for $u = 1$) one has $PSL(2,\mathbb{Z})_u = PSL(2,\mathbb{Z})$. It is known that the group B_3 is a central extension of $PSL(2,\mathbb{Z})$ of the centre consisting of the elements

$$(\hat{\sigma}_1\hat{\sigma}_2)^{3\lambda} = (\hat{\sigma}_2\hat{\sigma}_1)^{3\lambda} = (\hat{\sigma}_1\hat{\sigma}_2\hat{\sigma}_1)^{2\lambda} = (\hat{\sigma}_2\hat{\sigma}_1\hat{\sigma}_2)^{2\lambda} = \begin{pmatrix} t^{3\lambda} & 0 \\ 0 & t^{3\lambda} \end{pmatrix} \qquad \forall\lambda \in \mathbb{Z} \tag{9}$$

(let us note that the centre is isomorphic to $\mathbb{Z}$).

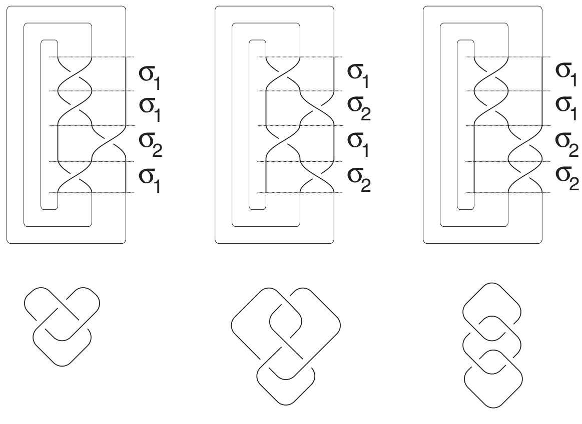

Figure 1. Some examples of closed braids and the corresponding links.

Recall that graphically, each word of B_3 corresponds to a particular three-strand braid, going from the top downwards, see figure 1. A closed braid is obtained by gluing the 'top' and 'bottom' free ends on a cylinder. Any closed braid defines a link (a knot is a particular case). However the correspondence between braids and links is not mutually single valued and each link can be represented by an infinite number of different braids (see [21, 22]). The irreducible length of a braid gives nevertheless a rich characteristic of the link complexity.

There exists extensive literature on the general properties of braid groups, see [21]; for the past works on the normal forms of words, we shall quote [23].

Any element of the group $G = \{PSL(2, \mathbb{Z}), H_q, B_3\}$ is defined by a word in the alphabets of the corresponding letters (generators):

- $\{S, T, T^{-1}\}$ or $\{a_2, b_3, b_3^{-1}\}$ for $PSL(2, \mathbb{Z})$
- $\{S, T_q, T_q^{-1}\}$ or $\{a_2, b_q, b_q^{-1}\}$ for H_q
- $\{\sigma_1, \sigma_2, \sigma_1^{-1}, \sigma_2^{-1}\}$ or $\{\tilde{a}, \tilde{a}^{-1}, \tilde{b}, \tilde{b}^{-1}\}$ for B_3.

We denote by w_n a word corresponding to a given sequence of letters of length n, and by $L^G(w_n)$—the irreducible length in the metric of words (the superscript G is used only when it is necessary), or in other terms the minimal number of generators necessary to build w_n. The irreducible length can be also viewed as a distance from the unity on the Cayley graph of the group G. Note that $L^G(w)$ depends on the set of generators we consider.

2.2. *Cayley graphs*

The modular group $PSL(2, \mathbb{Z})$ is a particular case of the Hecke group H_q at $q = 3$. We consider the more general case of the Cayley graphs of the groups H_q for $q = 3, 4, \ldots$. The Cayley graph of B_3 will be constructed afterwards. We investigate in this section only the abstract presentation of the groups in terms of commutation relations and do not pay attention to any matrix representation. We recall that the Cayley graph of a group G is the graph whose vertices are labelled by group elements, and whose links are as follows: w and w' are linked if and only if there exists a generator g such that $w' = wg$. Following this rule, we can easily construct the Cayley graph $\mathcal{G}_q$ of the group H_q represented by $\{a_2, b_q, b_q^{-1}\}$.

For any finite values of q (q is integer) the graph $\mathcal{G}_q$ has local q-cycles (because b_q is of order q), while the corresponding 'dual', or 'backbone', see figure 2, graph is the tree graph $\mathbb{T}_q$, which is precisely the graph of F_q. This is due to the free product structure of $H_q \sim \mathbb{Z}_2 \star \mathbb{Z}_q$

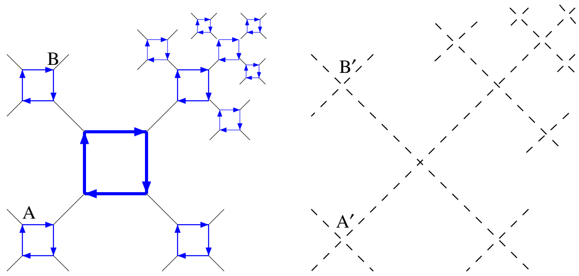

Figure 2. Cayley graph of H_q, with $q = 4$. Arrows correspond to generator $b_q = ST_q$, thin lines to generator $a_2 = S = S^{-1}$. The dashed graph is the corresponding backbone graph, a tree $\mathbb{T}_q$, graph of F_q. The distance d between A and B is $d = 5$, while the distance k (corresponding to the distance between A′ and B′) is $k = 2$.

(see explanations below). The graph $\mathcal{G}_q$ is shown in figure 2, where the backbone graph is marked by a dotted line.

3. Diffusion on graphs

In this section, we investigate some statistical properties of random walks on the groups introduced above, using their Cayley graphs. In particular, we consider *simple random walks*, that are walks of nearest neighbour type with symmetric transition probabilities.

3.1. Random walk on PSL(2, $\mathbb{Z}$)

Let us consider the free product groups of the form $\mathbb{Z}_2 \star \mathbb{Z}_q$ (isomorphic to H_q) in the framing which uses the generators a_2 and b_q of orders 2 and q, respectively. Graphs of such groups are shown in figure 2. We define two different 'metrics' of words on these graphs. The first metric is associated with the geodesic distance d on the graph—the minimal number of steps between two points belonging to the graph, and the second metric is associated with the geodesic distance k (called later the 'generation') on the backbone graph $\mathbb{T}_q$. Our goal is to compute the probability $P_q(d, n)$ of being at a distance d from the initial point (the root of the graph) after n random steps. The probability $\bar{P}_q(k, n)$ of being on the backbone graph at a generation k from a root point after n random steps will also be of use.

First of all we compute $P_3(d, n)$ and $\bar{P}_3(k, n)$ for the case of $PSL(2, \mathbb{Z})$. In this case the graph structure ensures the relation

$$|d - 2k| \leqslant 1.$$

Therefore we can consider only $\bar{P}_3(k, n)$. Write

$$\bar{P}_3(k, n) = \bar{P}_3^i(k, n) + \bar{P}_3^o(k, n)$$

distinguishing for an elementary triangular cell located at generation k the vertex closest to the root (corresponding to $\bar{P}_3^i(k, n)$) and the two others (corresponding to $\bar{P}_3^o(k, n)$) (see figure 3).

A direct enumeration gives the following master equation for $k \geqslant 2$:

$$\begin{cases} \bar{P}_3^i(k, n+1) = \frac{1}{3}\left(\bar{P}_3^o(k, n) + \bar{P}_3^o(k-1, n)\right) \\ \bar{P}_3^o(k, n+1) = \frac{2}{3}\bar{P}_3^i(k, n) + \frac{1}{3}\left(\bar{P}_3^o(k, n) + \bar{P}_3^i(k+1, n)\right) \end{cases} \qquad (10)$$

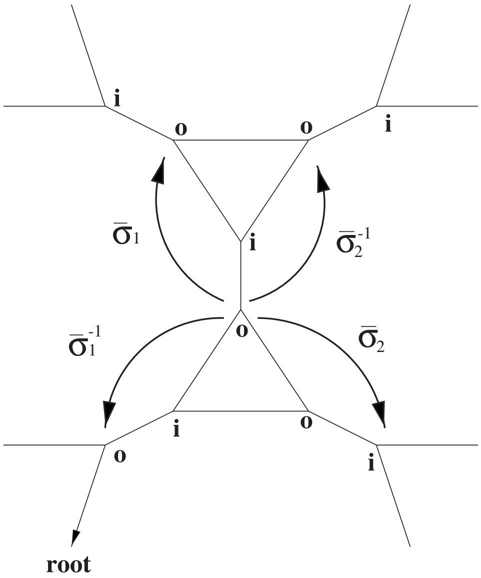

Figure 3. Random walk on $PSL(2, \mathbb{Z})$ in terms of generators $\bar{\sigma}_i$. Vertices of type 'i' and 'o' are shown.

with initial conditions of the form

$$\begin{cases} \bar{P}_3^i(k,0) = \alpha\delta_{k,0} \\ \bar{P}_3^o(k,0) = (1-\alpha)\delta_{k,0} \end{cases} \tag{11}$$

where α is an arbitrary parameter fixing the initial condition and lying in the interval $0 \leqslant \alpha \leqslant 1$. We are seeking the asymptotic $(1 \ll k \leqslant n)$ solution to (10) near the maximum of the probability distribution, and therefore will not take into account the specific form of the boundary condition.

Define the Laplace–Fourier transform:

$$Q^{i,o}(x,s) = \mathcal{T}\left[\bar{P}_3^{i,o}\right] = \sum_{n=0}^{\infty} s^n \sum_{k=-\infty}^{+\infty} \mathrm{e}^{\mathrm{i}kx}\, \bar{P}_3^{i,o}(k,n) \tag{12}$$

whose inverse can be written in the form

$$\bar{P}_3^{i,o}(k,n) = \frac{1}{4\mathrm{i}\pi^2} \oint \frac{\mathrm{d}s}{s^{n+1}} \int_{-\pi}^{\pi} \mathrm{e}^{-\mathrm{i}kx}\, Q^{i,o}(x,s)\,\mathrm{d}x. \tag{13}$$

One straightforwardly obtains the following algebraic system of linear equations:

$$\begin{cases} Q^i(x,s) - \dfrac{s}{3}(1+\mathrm{e}^{\mathrm{i}x})Q^o(x,s) = \alpha \\ -\dfrac{s}{3}(2+\mathrm{e}^{-\mathrm{i}x})Q^i(x,s) + \left(1-\dfrac{s}{3}\right)Q^o(x,s) = 1-\alpha \end{cases} \tag{14}$$

which determines the function $Q^{i,o}(x,s)$:

$$Q^{i,o}(x,s) = \frac{a_\alpha^{i,o}(x) + b_\alpha^{i,o}(x)s}{s^2 + \dfrac{3}{3+\mathrm{e}^{-\mathrm{i}x}+2\,\mathrm{e}^{\mathrm{i}x}}s - \dfrac{9}{3+\mathrm{e}^{-\mathrm{i}x}+2\,\mathrm{e}^{\mathrm{i}x}}} = \frac{a_\alpha^{i,o}(x) + b_\alpha^{i,o}(x)s}{p(x,s)} \tag{15}$$

where

$$\begin{cases} a^i_\alpha(x) = \dfrac{9\alpha}{3+\mathrm{e}^{-\mathrm{i}x}+2\,\mathrm{e}^{\mathrm{i}x}} \\ a^o_\alpha(x) = \dfrac{9(1-\alpha)}{3+\mathrm{e}^{-\mathrm{i}x}+2\,\mathrm{e}^{\mathrm{i}x}} \\ b^i_\alpha(x) = \dfrac{3(1-2\alpha+(1-\alpha)\mathrm{e}^{\mathrm{i}x})}{3+\mathrm{e}^{-\mathrm{i}x}+2\,\mathrm{e}^{\mathrm{i}x}} \\ b^o_\alpha(x) = \dfrac{3\alpha(2+\mathrm{e}^{-\mathrm{i}x})}{3+\mathrm{e}^{-\mathrm{i}x}+2\,\mathrm{e}^{\mathrm{i}x}}. \end{cases} \tag{16}$$

Denote by $s_\pm(x)$ the roots of $p(x,s)$. Using (13) one can rewrite

$$\bar{P}_3^{i,o}(k,n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{d}x\,\frac{\mathrm{e}^{-\mathrm{i}kx}}{s_+ - s_-}\left[a^{i,o}_\alpha(x)\left(\frac{1}{s_+^{n+1}} - \frac{1}{s_-^{n+1}}\right) + b^{i,o}_\alpha(x)\left(\frac{1}{s_+^{n}} - \frac{1}{s_-^{n}}\right)\right]. \tag{17}$$

We are interested in the $n, k \gg 1$ regime, and therefore consider the integrand in (17) for $x \to 0$. Here we expose the second-order computation, keeping in mind that any order can be reached in the same way. With

$$\begin{cases} s_+ = -\dfrac{3}{2} + \dfrac{3\,\mathrm{i}x}{20} - \dfrac{51x^2}{250} + O(x^3) \\ s_- = 1 - \dfrac{\mathrm{i}x}{15} + \dfrac{209x^2}{2250} + O(x^3) \end{cases} \tag{18}$$

one gets

$$\begin{aligned} \bar{P}_3(k,n) &\approx \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{d}x\,\mathrm{e}^{-\mathrm{i}kx}\left(1 - \frac{\mathrm{i}x}{15} + \frac{209x^2}{2250}\right)^{-n} \\ &\approx \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}x\,\exp\left[-\mathrm{i}kx - n\left(\frac{107x^2}{1125} - \frac{\mathrm{i}x}{15}\right)\right] \\ &\approx \frac{A}{n^{1/2}}\exp\left[-\frac{1125\left(k-\frac{n}{15}\right)^2}{428n}\right] \end{aligned} \tag{19}$$

where A is the normalization constant. Let us stress once again that expression (19) defines the probability of having the end-to-end distance in k steps along the backbone on the Cayley graph of the group $PSL(2,\mathbb{Z})$ for the n-step random walk; this result is valid *only* near the maximum (in k) of the distribution function $\bar{P}_3(k,n)$. The physical sense of (19) is very transparent: the effective curvature of the target space produces the 'driving force' (in the direction *from* the origin) for the random walk. Hence, the resulting distribution is transformed into the one-dimensional Gaussian function with the shift in the exponent.

Expression (19) allows one to compute the limiting value of the normalized drift $\bar{l}_3$

$$\bar{l}_3 = \lim_{n\to\infty}\frac{\langle k\rangle_3}{n}$$

on the backbone graph $\mathbb{T}_3$, where

$$\langle k\rangle_3 = \int_{-\infty}^{\infty} k\bar{P}_3(k,n)\,\mathrm{d}k = \frac{n}{15}$$

and, hence, the drift l_3

$$l_3 = \lim_{n\to\infty}\frac{\langle d\rangle_3}{n} = 2\bar{l}_3 = \frac{2}{15}$$

on the graph H_3.

This formalism can be generalized to the case of H_q, $q \geqslant 4$ as shown in appendix B, the drift in this case being accessible only numerically.

3.2. Random walk on B_3: drift and return probability

3.2.1. Analytic results. We now turn to the braid group B_3 and in particular show that some statistical characteristics of random processes on B_3 have the same asymptotic behaviour as that on $PSL(2, \mathbb{Z})$. The key point is that B_3 is a central extension of $PSL(2, \mathbb{Z})$. Let us recall that the centre Z of B_3, generated by $\tilde{a}^2 = \tilde{b}^3$, is isomorphic to $\mathbb{Z}$. This is a general feature of braid groups: the centre is generated by the square of the longest Coxeter element (or half-twist) denoted by Δ. In the case of B_3, $\Delta = \tilde{a}$. We denote by π the canonical quotient map

$$\pi : B_3 \longrightarrow B_3/Z \sim PSL(2, \mathbb{Z}). \tag{20}$$

One then has

$$\begin{cases} \pi(\sigma_1) = \bar{\sigma}_1 = a_2 b_3^{-1} \\ \pi(\sigma_2) = \bar{\sigma}_2 = b_3^{-1} a_2 \\ \pi(\tilde{a}) = a_2 \\ \pi(\tilde{b}) = b_3 \end{cases} \tag{21}$$

where a_2 and b_3 are defined in (1).

A natural representation of the Cayley graph of B_3 is three dimensional. As shown in figure 4, the map π can then be viewed as a projection from 3D to 2D.

Consider now an n-letter random word w_n written in terms of the generators of the group B_3:

$$w_n = \prod_{i=1}^{n} \sigma_{r_i}$$

where we have set $\sigma_{r_i}^{-1} = \sigma_{-r_i}$ and indices r_i are uniformly distributed in $\{-2, -1, 1, 2\}$. We recall that $L^{B_3}(w_n)$ is the irreducible length of w_n. It is evident that

$$L^{B_3}(w_n) \geqslant L^{PSL(2,\mathbb{Z})}(\pi(w_n)). \tag{22}$$

Keeping in mind the geometrical interpretation of B_3 shown in figure 4, we can easily derive equation (22) from a triangular inequality.

Consider now the irreducible decomposition in $PSL(2, \mathbb{Z})$:

$$\pi(w_n) = \prod_{i=1}^{L^{PSL(2,\mathbb{Z})}(\pi(w_n))} \bar{\sigma}_{r_i'}. \tag{23}$$

The linear asymptotic of $\langle L^{PSL(2,\mathbb{Z})}(\pi(w_n))\rangle$ for $n \gg 1$ is computed in appendix A for the group $PSL(2, \mathbb{Z})$. From (23) and the definition of the quotient map, we get

$$w_n = \Delta^{2f(n)} \prod_{i=1}^{L^{PSL(2,\mathbb{Z})}(\pi(w_n))} \sigma_{r_i'}. \tag{24}$$

As Δ^2 is a six-letter word in the alphabet $\left\{\sigma_1, \ldots, \sigma_2^{-1}\right\}$ (see equation (9)), the following condition on $f(n)$ holds:

$$|f(n)| \leqslant \frac{n}{6}.$$

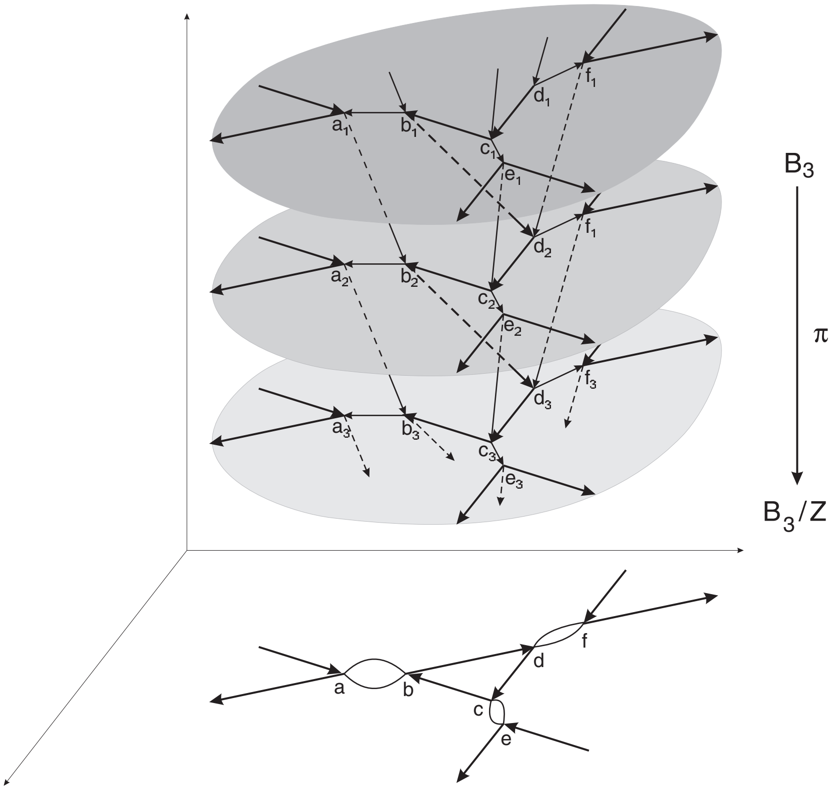

Figure 4. B_3 Cayley graph and its projection ($PSL(2, \mathbb{Z})$). Thin arrows correspond to $\tilde{a}$, thick ones to $\tilde{b}$. Note that $\pi(\alpha_i) = \alpha$, $\pi(\beta_i) = \beta$ and so on. Recall that a_2 has to be identified with a_2^{-1}.

Hence, the irreducible length of the word w_n can be estimated from above

$$L^{B_3}(w_n) \leqslant 6|f(n)| + L^{PSL(2,\mathbb{Z})}(\pi(w_n)). \tag{25}$$

Let us show now that the process $f(n)$ is such that

$$\langle |f(n)| \rangle = O(\sqrt{n}). \tag{26}$$

One can use the methods of semi-Markov chains. These rest on the fact that an increment of f is a regenerative event and the times between successive increments are i.i.d. One then uses the following:

1. The symmetry of the process implies that the words Δ^2 and Δ^{-2} appear with the same probability, which gives $\langle f(n) \rangle = 0$.
2. The increment $|f(n+1) - f(n)|$ is bounded from above by some constant.

Thus, the law of large numbers gives (26). This, together with (22) and (25), allows one to write

$$\frac{L^{PSL(2,\mathbb{Z})}(\pi(w_n))}{n} \leqslant \frac{L^{B_3}(w_n)}{n} \leqslant \frac{L^{PSL(2,\mathbb{Z})}(\pi(w_n))}{n} + O\left(\frac{1}{\sqrt{n}}\right). \tag{27}$$

Using the result $\lim_{n\to\infty} L^{PSL(2,\mathbb{Z})}(\pi(w_n))/n = \frac{1}{4}$ obtained in appendix A for the symmetric random walk on $PSL(2,\mathbb{Z})$, one arrives for generators $\{\bar\sigma_1, \ldots, \bar\sigma_2^{-1}\}$ at the following asymptotic expression:

$$l_{B_3} = \lim_{n\to\infty} \frac{L^{B_3}(w_n)}{n} = \frac{1}{4}. \tag{28}$$

3.2.2. Statistics of loops on B_3: return probability for 'magnetic' random walks. The investigation carried out above shows that if a random walk on the group B_3 ends in the centre $Z(B_3)$ (such a walk we shall call a Z-walk), it can be regarded as a closed 'magnetic' random walk on $PSL(2,\mathbb{Z})$. Namely, if one inserts in each elementary cell of the hyperbolic lattice shown in figure 4 a 'magnetic flux' h and denotes by Φ the total flux through a closed path on $PSL(2,\mathbb{Z})$, then any word w_n^Z corresponding to a Z-walk on B_3 can be written as

$$w_n^Z = \Delta^{2\Phi/h}.$$

In other words, the group B_3, being the central extension of $PSL(2,\mathbb{Z})$, gives rise to a fibre bundle over $PSL(2,\mathbb{Z})$ such that every full turn around the elementary cell leads to another sheet of the Riemann surface of $PSL(2,\mathbb{Z})$. The outcome of this construction is that Z-walks on B_3 can be decomposed into a product of elementary full turns around cells (this is due to the tree structure of the backbone).

Let $u_n(\Phi)$ be the probability that a closed n-step loop on a graph $PSL(2,\mathbb{Z})$ carries a flux Φ. The function $u_n(\Phi)$ is of major interest for us, especially because at $\Phi = 0$ it is connected to the probability of finding a *trivial braid* (i.e. completely reducible word) from a random braid of initial length n (see equations (35)–(37)).

First of all we compute $u_n^a(\Phi)$ for a walk on the group $PSL(2,\mathbb{Z})$ with local passages in the basis $\{a_2, b_3, a_2^{-1}, b_3^{-1}\}$ (let us stress that for magnetic walks one has to distinguish 'artificially' between steps a_2 and a_2^{-1}, in order to assign an area to loops a_2^2). Denote by $\#a_2, \#a_2^{-1}, \#b_3, \#b_3^{-1}$, respectively, the total number of steps $a_2, a_2^{-1}, b_3, b_3^{-1}$ in a given closed path on $PSL(2,\mathbb{Z})$. The flux Φ can be decomposed as follows:

$$\Phi = \frac{h}{6}\left(3\left(\#a_2 - \#a_2^{-1}\right) + 2\left(\#b_3 - \#b_3^{-1}\right)\right). \tag{29}$$

Recall that we consider an n-step process on $PSL(2,\mathbb{Z})$, conditioned by the fact that the path is closed (i.e. returns to the origin). Following (29), we introduce a simultaneous process Φ_i (with $\Phi_0 = 0$) such that

$$\Phi_{i+1} = \Phi_i + \phi_{i+1} \tag{30}$$

where

$$\phi_i = \begin{cases} \pm h/2 & \text{if the step is } a_2^{\pm 1} \\ \pm h/3 & \text{if the step is } b_3^{\pm 1}. \end{cases} \tag{31}$$

Evidently the final value Φ_n gives the total flux Φ through the closed path. This process is as follows:

1. Note that on $PSL(2,\mathbb{Z})$ we have $a_2^{-1} = a_2$, and therefore $p(\phi_i = h/2) = p(\phi_i = -h/2)$.
2. The sign of the magnetic field can be arbitrarily changed, hence $p(\phi_i = h/3) = p(\phi_i = -h/3)$ (i.e. positive, b_3^3, and negative, b_3^{-3}, elementary turns are equidistributed for closed as well as for open paths).
3. We show that at least asymptotically the process Φ_i does not depend on the condition that the path is closed. The closure condition on $PSL(2,\mathbb{Z})$ is a condition on the irreducible length of words. The irreducible forms on $PSL(2,\mathbb{Z})$ are exactly the words of the form

$a^{\pm 1}b^{\pm 1}a^{\pm 1}b^{\pm 1}a^{\pm 1}\ldots$. The relative weight of $a^{\pm 1}$ and $b^{\pm 1}$ in this irreducible form is 1, just as for the 'free' (without any condition) process. It means that even if the condition that the path is closed is imposed, one still has $p(\phi_i = h/2) = p(\phi_i = h/3)$.

The process Φ_i is then a classical one-dimensional random walk, and therefore for n large, one has

$$u_n^a(\Phi) = \frac{1}{h\sigma_a\sqrt{2\pi n}} \exp\left(-\frac{(\Phi/h)^2}{2n\sigma_a^2}\right) \tag{32}$$

where $\sigma_a^2 = \frac{1}{2}\left(\frac{1}{4} + \frac{1}{9}\right) = \frac{13}{72}$.

Returning to the random walk on the braid group in the standard framing $\{\sigma_1, \sigma_2, \sigma_1^{-1}, \sigma_2^{-1}\}$, we can compute the distribution $u_n^\sigma(\Phi)$ for a random process on $\{\bar{\sigma}_1, \bar{\sigma}_2, \bar{\sigma}_1^{-1}, \bar{\sigma}_2^{-1}\}$. Modifying slightly the derivation carried out above, one obtains that the corresponding process ϕ_i is such that $p(\phi_i = h/6) = p(\phi_i = -h/6) = 1/2$. This yields

$$u_n^\sigma(\Phi) = \frac{1}{h\sigma_\sigma\sqrt{2\pi n}} \exp\left(-\frac{(\Phi/h)^2}{2n\sigma_\sigma^2}\right) \tag{33}$$

with $\sigma_\sigma^2 = \frac{1}{36}$.

This result seems to be important in the context of lattice random walks in a transversal magnetic field [25] which has relations with the Harper–Hofstadter problem (see [26] for a review) in hyperbolic geometry. Note that $\Phi/h \propto \mathcal{A}$ counts the algebraic area $\mathcal{A}$ enclosed by the random path on the graph of the group $PSL(2, \mathbb{Z})$. It is interesting to compare the result (33) to the exact solution of the area problem in the continuous case, derived in [27]. The probability distribution in this case reads

$$p_t(A) \propto \frac{\exp(-A^2/2t)}{\cosh^2(\pi A/t)} \tag{34}$$

where A is the dimensionless hyperbolic area and t is the time. For long trajectories ($t \gg 1$), one can check that the Gaussian behaviour (33) is recovered. For short trajectories ($t \ll 1$) one recovers the flat space limit, unreachable in our model because the length scale is fixed by the curvature. This asymptotic agreement between the continuous and discrete models is an important fact, though intuitively expectable.

The decomposition introduced above allows us to compute the return probability, i.e. the probability $p(w_n = I_d) = p_r(n)$ of obtaining a 'trivial' braid after n random elementary moves. Using (24) the condition $w_n = I_d$ is equivalent to the conditions

$$L^{PSL(2,\mathbb{Z})}(\pi(w_n)) = 0 \qquad \text{and} \qquad f(n) = 0.$$

Denote

$$p\{L^{PSL(2,\mathbb{Z})}(\pi(w_n)) = 0\} = p_r^\pi(n)$$

and

$$p\{f(n) = 0 \,\text{knowing that}\, L^{PSL(2,\mathbb{Z})}(\pi(w_n)) = 0\} = p_r^c(n).$$

The probabilities $p_r^\pi(n)$ and $p_r^c(n)$ obey the following relation:

$$p_r(n) = p_r^\pi(n) p_r^c(n) \tag{35}$$

where $p_r^\pi(n)$ is computed in appendix A and $p_r^c(n)$ can be re-expressed in the following way:

$$p_r^c(n) = h u_n^\sigma(0). \tag{36}$$

Collecting (35)–(36) we arrive at the final expression for $p_r(n)$:

$$p_r(n) = \frac{C}{\sigma_\sigma\sqrt{2\pi}} \frac{\lambda^n}{n^2} \tag{37}$$

where $\lambda = \frac{2\sqrt{2}+1}{4}$ and $C = \frac{9+4\sqrt{2}}{7\pi}$ (these constants are computed in appendix A).

3.2.3. Numerical results. So far there is no constructive algorithm to find the reduced form of words of B_3 for generators σ_i. The existence of an algorithm depends crucially on the set of generators we choose. Indeed, it is shown in [28] that computing the length in terms of generators σ_i of a braid in B_n is an NP-complete problem. Let us mention however that braid groups are 'biautomatic' (see [29]) which basically means that there exists a set of generators for which the reduced words are exactly known. This allows in particular to solve the word enumeration problem and to implement methods which can compare two different braids in a polynomial time (see [30]). In our case of the simplest nontrivial group B_3 we tried a random reduction procedure, but it converges only in exponential time. Since our analytical results are obtained in the regime ($n \gg 1$), the numerical simulations give no additional information.

4. Diffusion on hyperbolic lattices: traces and Lyapunov exponents

We consider the representation of dimension 2 of the groups introduced above, and investigate their action on the hyperbolic Poincaré plane $\mathcal{H} = \{z, \operatorname{Im} z > 0\}$. Namely, we consider the following fractional-linear transforms:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : z \to \frac{az+b}{cz+d}. \tag{38}$$

We recall that $PSL(2, \mathbb{R})$ is the group of orientation preserving isometries of $\mathcal{H}$. The groups $PSL(2, \mathbb{Z})$, $PSL(2, \mathbb{Z})_u$, H_q, F_n admit representations as subgroups of $PSL(2, \mathbb{R})$ and their Cayley graphs (considered in the previous section) are now viewed as isometric lattices embedded into $\mathcal{H}$. Now one can investigate their metric properties using the natural hyperbolic (geodesic) distance in $\mathcal{H}$. We define the lattices under consideration in the same way as we have defined the Cayley graphs:

- We construct the set of all possible orbits of a given root point (we choose the point $i = (0, 1)$ for convenience) under the action of the group.
- We denote by $\delta(w_n)$ the hyperbolic distance $\delta(i, w_n(i))$ between i and $w_n(i)$.
- We call 'lattices' (keeping in mind that the precise definition of a lattice, not necessary here, is a discrete subgroup with cofinite volume) the Cayley graphs of the groups involved here because of two important features:
  - they are discrete subgroups of $PSL(2, \mathbb{R})$, the group of motion of the hyperbolic 2-space. Hyperbolic distance is a pair–point invariant, that is $\delta(i, w(i)) = \delta(\gamma i, w\gamma(i))$, which justifies the term isometric;
  - they have the property of so-called lattice groups: they have no points of accumulation (for the topology of $\mathcal{H}$).

Let us add that the above description is based on the well-known results on Fuchsian group theory (another designation of discrete subgroups of $PSL(2, \mathbb{R})$, see [11, 31]). The properties of a Fuchsian group G depend strongly on the fundamental domain of G, which is a minimal set of points generating $\mathcal{H}$ under the action of G. We recall that the fundamental domain of the Hecke group is the circular triangle with angles $\left\{0, \frac{\pi}{q}, \frac{\pi}{q}\right\}$ (see figure 5 for H_3).

It can be shown that the fundamental domain of F_n is a zero-angled n-gon. Our contribution to this subject concerns the construction of the fundamental domain of the deformed group $PSL(2, \mathbb{Z})_u$ (figure 6).

This construction is based on the general method exposed in [11]. The outline is as follows. We first find the fixed point i/u of S_u, and $x_0 = 1/(1 - u^2)$ of T_u. We then draw the only geodesic through i/u which intersects its images by T_u and T_u^{-1} with angle $\pi/3$. Circles of centre x_0 passing these intersections complete the construction. Let us note that

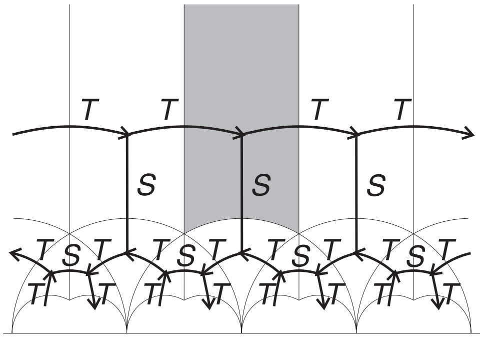

Figure 5. Fundamental domain of $PSL(2, \mathbb{Z})$.

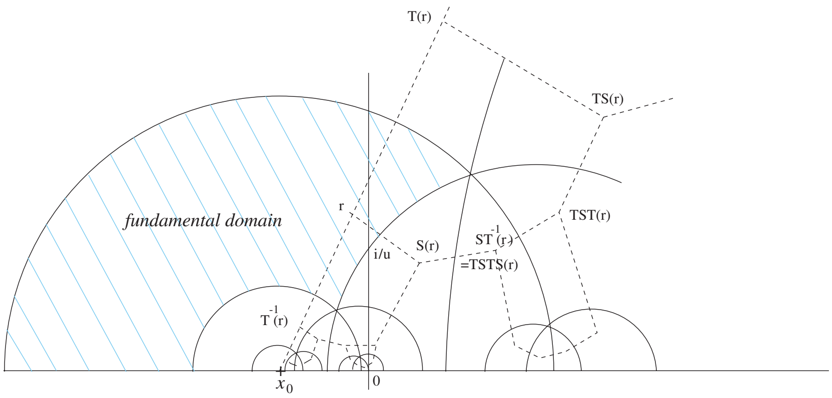

Figure 6. Fundamental domain of $PSL(2, \mathbb{Z})_u$, for $u = 1.2$.

the topology of the Cayley graph obtained in this way does not depend on u. Recall that only commutation relations, independent of u, set the topological structure of the Cayley graph. Only the metric properties are affected by u. In particular, the area of the fundamental domain is finite only for $u = 1$. The group is then said to be of type I in the classification of Fuchsian groups. For $u \neq 1$ it is of type II. It means that the corresponding monodromy problems are deeply different (see [32]). Solving the monodromy problem is an important issue since it allows us to get the conformal transform that maps the fundamental domain onto $\mathcal{H}$. To our knowledge the problem is solved only for $u = 1$. We have therefore to content ourselves with an existence theorem in the general case. The existence of such a transform allows us to define a map f_u from the fundamental domain of $PSL(2, \mathbb{Z})_u$ to the fundamental domain of $PSL(2, \mathbb{Z})$. The action of $PSL(2, \mathbb{Z})_u$ on $\mathcal{H}$ is in this sense conjugate to the action of $PSL(2, \mathbb{Z})$:

$$\forall \omega_u \in PSL(2, \mathbb{Z})_u \qquad \omega_u(z) = f_u^{-1} \circ \omega_{u=1} \circ f_u(z). \tag{39}$$

The dependence on the parameter u is clearly expressed in this way.

4.1. Analytic results

Let us return to the definition of the model and recall that the groups under consideration act in the hyperbolic Poincaré upper half-plane $\mathcal{H} = \{z \in \mathbb{C}, \mathrm{Im}(z) > 0\}$ by fractional–linear transforms[4]. The matrix representation of the generators (denoted by h_i, $1 \leqslant i \leqslant n_g$) of the different groups has been given in section 2.

Choosing the point $(x_0, y_0) = (0, i)$ as the tree root, see figure 5, we associate any vertex on the lattice with an element $w_n = \prod_{k=1}^{n} h_{\alpha_k}$ where $1 \leqslant \alpha_k \leqslant n_g$ and w_n is parametrized by its complex coordinates $z_n = w_n(i)$ in the hyperbolic plane.

Strictly speaking $\mathcal{H}$ should be identified with $SL(2, \mathbb{R})/SO(2)$; we here identify an element with its class of equivalence of $SO(2)$. The following identity holds (see [11, 33]):

$$2\cosh(\delta(w_n)) = \mathrm{Tr}\big(w_n w_n^{\dagger}\big) \tag{40}$$

where the dagger denotes transposition.

We are interested in the distribution function $P_n(\delta)$, and therefore have to look for the distribution of traces of matrices w_n. The method described hereafter involves mainly the results of [34]. The outline of our approach is as follows. We study the behaviour of the random matrix w_n, generated by a Markov chain (which must fulfil ergodicity properties, see [35]) defined as follows:

$$w_{n+1} = w_n h_{\alpha_{n+1}} \qquad (\alpha_{n+1} = i, 1 \leqslant i \leqslant n_g) \quad \text{with probability} \quad \frac{1}{n_g}. \tag{41}$$

We use the standard methods of random matrices and consider the entries of the 2×2 matrix w_n as a 4-vector $\mathcal{V}_n$. A given transformation $w_{n+1} = w_n h_\alpha$ now reads

$$\mathcal{V}_{n+1} = \begin{pmatrix} h_\alpha^{\dagger} & 0 \\ 0 & h_\alpha^{\dagger} \end{pmatrix} \mathcal{V}_n. \tag{42}$$

This block-diagonal form allows us to study one of the two 2-vectors composing $\mathcal{V}_n$, say v_n. Parametrizing $v_n = (\varrho_n \cos\theta_n, \varrho_n \sin\theta_n)$ and using the relation $\delta(w_n) \equiv \delta_n \simeq 2\ln\varrho_n$ valid for $n \gg 1$, one gets a recursion relation $v_{n+1} = h_\alpha^{\dagger} v_n$ in terms of hyperbolic distance δ_n:

$$\delta_{n+1} = \delta_n + \ln p_\alpha(\cos\theta) \tag{43}$$

where p_α is a second-order polynomial depending on the specific form of transition matrices h_α. While for the angles one gets straightforwardly

$$\cot\theta_{n+1} = h_\alpha(\cot\theta_n). \tag{44}$$

The action of h_α is fractional–linear.

One now has to study the invariant measure $\mu(\theta)$, giving the asymptotic probability of having $\theta_n = \theta$. Introducing $x = \cot\theta$, we are led to study the action of the group restricted on the real line parametrized by x. The statistical properties of μ have been discussed by Gutzwiller and Mandelbrot [36] in the case of the free group Λ. An alternative, put forward in [34], is to define $\mu(x)$ as the limit of the following recursion relation:

$$\mu^{(n+1)}(x) = \frac{1}{n_g} \sum_{\alpha=1}^{n_g} \mu^{(n)}(h_\alpha(x)) \left| \frac{\mathrm{d}h_\alpha(x)}{\mathrm{d}x} \right|. \tag{45}$$

The convergence $\mu^{(n)}(x) \to \mu(x)$ for $n \to \infty$ depends on the properties of the functional transform (45). One should first mention that according to Furstenberg theory [37], this convergence to the invariant distribution $\mu(x)$ holds in the weak sense. Our point here is to give a numerically accessible definition of $\mu(x)$, which requires the convergence in distribution. It has been successfully checked numerically by comparing to the direct sampling of different orbits that this limit exists independently of initial conditions. Despite the absence of a

[4] It is convenient first to define the representation in the Poincaré upper half-plane and then use the conformal transform to the unit disc.

rigorous proof, we claim that μ is defined with no ambiguity by (45). This enables us to compute the desired distribution $P_n(\delta)$. The crucial point required for the convergence of $\mu^{(n)}$ to the invariant distribution, is the existence of ergodic properties of θ_n. It means that for $n \gg 1$, the distribution of θ_n is given by $\mu(\theta)$, independently of n and initial conditions (see [35] for precise definitions). We introduce the generating function for (43); due to the Markovian structure of (43), we can perform the averaging:

$$\langle \mathrm{e}^{\mathrm{i}k\delta_{n+1}} \rangle = \langle \mathrm{e}^{\mathrm{i}k\delta_n} \rangle \langle [p_\alpha(\cos\theta)]^{\mathrm{i}k} \rangle. \tag{46}$$

Thus we obtain

$$\langle \mathrm{e}^{\mathrm{i}k\delta_n} \rangle = \left[\frac{1}{n_g} \sum_{\alpha=1}^{n_g} \int_{-\pi/2}^{\pi/2} \mathrm{d}\theta\, \mu(\theta) [p_\alpha(\cos\theta)]^{\mathrm{i}k} \right]^n. \tag{47}$$

This form suggests that for n large, $P_n(\delta)$ satisfies a central limit theorem. Indeed such a theorem exists (see [38, 39]) for Markovian multiplicative processes (and in particular for random matrices, see [40]) provided that the phase space is ergodic. We are then led to compute only the first two moments (the first one being the Lyapunov exponent), which gives us

$$\gamma_1 = \lim_{n\to\infty} \frac{\langle\delta\rangle}{n} = \frac{1}{n_g} \sum_{\alpha=1}^{n_g} \int_{-\pi/2}^{\pi/2} \mathrm{d}\theta\, \mu(\theta) \ln p_\alpha(\cos\theta) \tag{48}$$

and

$$\gamma_2 = \frac{1}{n_g} \sum_{\alpha=1}^{n_g} \int_{-\pi/2}^{\pi/2} \mathrm{d}\theta\, \mu(\theta) \ln^2 p_\alpha(\cos\theta). \tag{49}$$

Hence we get for the dispersion:

$$\sigma^2 = \lim_{n\to\infty} \frac{\langle(\delta - \langle\delta\rangle)^2\rangle}{n} = \gamma_2 - \gamma_1^2. \tag{50}$$

4.2. Numerical results

We present in this section the numerical and semi-analytical results for the invariant measure μ and the Lyapunov exponent γ_1. Our main goal is to compare the approach developed here with the results following from the study of random walks on graphs (see section 3).

Let us call the *backbone subgroup* $\mathcal{B}(G)$ of the group G the subgroup of G whose Cayley graph is the backbone of the graph of G. It seems to be more instructive to rely on this purely geometrical characterization of $\mathcal{B}$ and to avoid a formal definition. Let us stress that $\mathcal{B}(G)$ is a free subgroup of G. One has for example $\mathcal{B}(F_n) = F_n$. Consider now the representation of F_q by q idempotent generators $g_1, \ldots, g_q$ with the following homomorphism Ψ:

$$\Psi : \begin{cases} F_q \longrightarrow H_q \\ g_i \longrightarrow b_q^{-i} a_2 b_q^i. \end{cases} \tag{51}$$

Due to the injectivity of Ψ, the following decomposition holds:

$$H_q = \bigcup_{i=1}^{q} b_q^i \Psi(F_q) \tag{52}$$

with

$$b_q^i \Psi(F_q) \bigcap b_q^j \Psi(F_q) = \emptyset \qquad \text{for} \quad i \neq j \tag{53}$$

Table 1. The drift in a hyperbolic metric and in a metric of words.

Group	Generators	Backbone subgroup, scale factor r	$r\gamma_1^s/\gamma_1^d$ numerical	$r\gamma_1^s/\gamma_1^d$ semi-analytical	$l \equiv \langle L^G\rangle/n\ (n \gg 1)$: graph approach
F_3	h_1, h_2, h_3	$F_3, 1$	0.3334	0.332	$1/3$
F_4	$h_1, h_2, h_1^{-1}, h_2^{-1}$	$F_4, 1$	0.501	0.503	$1/2$
H_3	a_2, b_3, b_3^{-1}	$F_3, 2$	0.1334	0.132	$2/15 = 0.133\ldots$
$PSL(2,\mathbb{Z})$	$\bar{\sigma}_1, \bar{\sigma}_2, \bar{\sigma}_1^{-1}, \bar{\sigma}_2^{-1}$	$F_3, 1$	0.2501	0.248	$1/4$

which means that the Cayley graph of H_q is the disjoint union of q trees $\mathbb{T}_q$. Thus we set $\mathcal{B}(H_q) = F_q$.

The scale factor r is the 'average' irreducible length of the generators of $\mathcal{B}(G)$ viewed as elements of G. In other words,

$$L^G(w) \sim r L^{\mathcal{B}(G)}(w) \tag{54}$$

for $w \in \mathcal{B}(G)$ with $L^G(w) \gg 1$. We have studied two different Markovian processes in the space $\mathcal{H}$ where the group G is isometrically embedded:

- simple random walks, characterized by the Lyapunov exponent γ_1^s;
- *directed* random walks (that are walks excluding two consecutive opposite steps) on the backbone subgroup $\mathcal{B}(G)$, characterized by the Lyapunov exponent γ_1^d.

By construction $\ell = \gamma_1^d/r$ is the average hyperbolic length of an elementary step on G. We conjecture that $\gamma_1^s/\ell \equiv r\gamma_1^s/\gamma_1^d$ gives *the number of steps to the origin* (normalized by n) on the graph G, i.e. the drift l:

$$l = r\frac{\gamma_1^s}{\gamma_1^d}. \tag{55}$$

Let us point out that this result links together two definitions of the 'drifts' for random walks on the groups G: the drift $l = \lim_{n\to\infty} \frac{\langle L^G(w)\rangle}{n}$ is defined on the graph in the metric of words while γ_1^s and γ_1^d are defined in terms of hyperbolic distances for an isometric embedding of G into $\mathcal{H}$ (the scale factor r depends only on the set of generators under consideration and is metric independent). Thus we claim

$$\langle L^G(w)\rangle = r\frac{\langle \ln \mathrm{Tr}(ww^\dagger)\rangle}{\gamma_1^d} \tag{56}$$

where a word w is identified with its matrix representation. We believe that equation (56) is of great importance, since it relates the properties of a group defined only through symbolic commutation relations to the geometrical properties of a given representation.

The stochastic average $\langle\cdots\rangle$ in (56) is necessary, to wash out purely geometrical effects such as multifractality investigated in [34]. (It corresponds to fluctuations of the hyperbolic distance for words of the same length on the backbone graph.) One has to stress that (56) holds due to a 'global' angular symmetry (see [41] for a precise definition of this symmetry for graphs) of both models; only the 'radial' part of the processes is considered, whereas the angular dependence is averaged. This has been checked numerically in the continuous case: generators have to be properly normalized, such that each elementary step should have the same hyperbolic length, ensuring angular symmetry, else the invariant measure μ fails to converge.

All results are summarized in table 1. The semi-analytical computations are based on numerical evaluation of γ_1 in (48).

5. Discussion and perspectives

We have presented at length in this work different aspects of random walks on a family of hyperbolic-like groups. The main attention has been paid to the computations of the *drift* and the *return probability* of Markov processes on the corresponding groups. Let us summarize the results obtained in the paper.

5.1. The drift and related problems

On the one hand, we studied the Cayley graphs of these groups, and briefly exposed the general methods of computing the transition probabilities for Markovian processes on these graphs. In particular we explicitly calculate the drift in different cases. As an application, we studied Markovian processes on the braid group B_3, and explicitly showed that the drift l_{B_3} for a symmetric random walk on this group tends at $n \to \infty$ to the drift of a process on the group $PSL(2, \mathbb{Z})$, which is found to be $\frac{1}{4}$. It means that a typical random braid of string length n can be realized on average by $\frac{n}{4}$ elementary moves. In particular this result shows that the random vortex lines (represented by random braids) are highly entangled. The 'complexity' η of the entangled state can be characterized by the drift l_{B_3} and hence $\eta \sim \frac{n}{4}$. This property has purely non-Abelian nature. Namely, when distinguishing the topological states of braided lines just counting the corresponding winding numbers between the strings, we arrive at the conclusion that the average braid complexity behaves as $\eta \sim \sqrt{n}$ which is an incorrect result.

On the other hand, we took advantage of the fact that the groups H_q and $PSL(2, \mathbb{Z})$ are subgroups of $PSL(2, \mathbb{R})$ and therefore act naturally in the hyperbolic plane $\mathcal{H}$. The Cayley graphs of these groups are then isometrically embedded in $\mathcal{H}$. Instead of the usual length in the metric of word, we could, thanks to this representation, use the metric structure of $\mathcal{H}$ and study the hyperbolic length of random elements of the group. This problem leads to the study of products of random matrices. The method described in [34] allows us to compute the probability distribution of the hyperbolic length and the corresponding Lyapunov exponents.

These two approaches are shown to be related by an equation (56). This result is a strong motivation for investigating further the geometric properties of hyperbolic groups in connection with other topological invariants. As an example we briefly mention the Alexander polynomials of knots.

The Alexander polynomial $\nabla_K(t)$ of a link K represented by a closed braid $w_n = \prod_{j=1}^{n} \sigma_{r_j}$ of length n is defined as follows:

$$(1 + t + t^2)\nabla_K(t) = \det\left[\prod_{j=1}^{n} \hat{\sigma}_{r_j} - \hat{I}\right] = \det\left[\prod_{j=1}^{n} \hat{\sigma}_{r_j}\right] + 1 - \mathrm{Tr}\left[\prod_{j=1}^{n} \hat{\sigma}_{r_j}\right] \tag{57}$$

where j runs 'along the braid', i.e. labels the number of used generators, the subscript $r_j \in \{-2, -1, 1, 2\}$ marks the set of braid generators (letters), with the prescription $\hat{\sigma}_i^{-1} = \hat{\sigma}_{-i}$ and $\hat{I}$ defines the 2×2 identity matrix. For long words ($n \gg 1$), the following asymptotic expression holds:

$$\mathrm{Tr}(w_n) \sim \left(\mathrm{Tr}\left(w_n w_n^{\dagger}\right)\right)^{1/2} \sim \mathrm{e}^{\delta(w_n)/2}. \tag{58}$$

One then has, with the parameter $u = \sqrt{-t}$ (recall that $\hat{\sigma}_i$ depends on u):

$$(1 - u^2 + u^4)\nabla_K(u) = u^{2p(w_n)} - u^{p(w_n)}\, \mathrm{e}^{\delta(w_n)/2} + 1 \tag{59}$$

with $p(w_n) = \#(+) - \#(-)$. In this regime the polynomial is therefore expressed only in terms of $p(w_n)$ and $\delta(w_n)$. The quantity $p(w_n)$ is a 'poor' invariant, in the sense that it takes the same value for a large number of links. In other words $p(w_n)$ is just the length of the element w_n projected onto $\mathbb{Z}$. Indeed there exists an obvious group homomorphism π_1 from B_3 to $\mathbb{Z}$ defined by $\pi_1(\sigma_i^{\pm 1}) = \pm 1$. All non-Abelian properties are lost by this invariant. The geometric invariant $\delta(w_n)$, described above, is much stronger. As we have shown, this invariant is related directly to the word length in the group $PSL(2,\mathbb{Z})$, which preserves the noncommutative structure of B_3 (recall that the random word length in B_3 has the same asymptotics as the word length in $PSL(2,\mathbb{Z})$). The information is nevertheless not redundant, because there is no nontrivial homomorphism from $PSL(2,\mathbb{Z})$ to $\mathbb{Z}$ (there is no finite order element in $\mathbb{Z}$). In particular, under the condition $\delta(w_n) = 0$ (Z-walks), p is an exact invariant having the sense of a winding number.

The form (59) seems in particular convenient for the possible problems of statistics of Alexander polynomials, since we know the statistics of both p and d. In particular, for a simple random walk, the typical Alexander polynomial for a long braid could be defined as $\bar{\nabla}_n(u)$:

$$(1 - u^2 + u^4)\bar{\nabla}_n(u) = 1 - \mathrm{e}^{n\gamma_1(u)/2} \tag{60}$$

where $\gamma_1(u)$ is the Lyapunov exponent of the random product of generators $\hat{\sigma}_i$.

5.2. *The return probability*

The graph approach and the introduction of 'magnetic walks' enabled us also to compute explicitly the return probability on B_3, i.e. the probability $p_r(n)$ of getting a trivial (completely reducible) braid from a random word of string length n. The result $p_r(n) \propto \lambda^n/n^2$, where $\lambda = \frac{2\sqrt{2}+1}{4} < 1$ (see equation (37)) reflects again the non-Abelian nature of the problem under consideration: the return probability in the commutative space has no exponential dependence on n.

Special attention should be paid to the difference in the return probability on the group $PSL(2,\mathbb{Z})$ and on the braid group B_3 (which is the central extension of $PSL(2,\mathbb{Z})$). The factorization of the random walk on the group B_3 into two parts: the diffusion along the graph $PSL(2,\mathbb{Z})$ and along the centre lead, to the fact that pre-exponential asymptotics of the corresponding return probabilities are different: $p_r(n) \propto \lambda^n/n^2$ for B_3 versus $p_r(n) \propto \lambda^n/n^{3/2}$ for $PSL(2,\mathbb{Z})$.

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Appendix A. Drift on *PSL* (2, $\mathbb{Z}$)

The aim of this appendix is to compute the drift of a random walk on $PSL(2,\mathbb{Z})$ in terms of generators $\bar{\sigma}_i$. We keep the notation of section 3 and proceed in the same way, noting that the process under consideration is no longer a simple random walk, but is described by the transitions shown in figure 3. A direct counting gives the following master equation for $k \geqslant 2$:

$$\begin{cases} \bar{P}_3^i(k,n+1) = \frac{1}{4}\big(\bar{P}_3^i(k+1,n) + 2\bar{P}_3^i(k-1,n) + \bar{P}_3^o(k-1,n)\big) \\ \bar{P}_3^o(k,n+1) = \frac{1}{4}\big(\bar{P}_3^o(k+1,n) + \bar{P}_3^i(k+1,n) + 2\bar{P}_3^o(k-1,n)\big) \end{cases} \tag{A1}$$

with initial conditions of the form

$$\begin{cases} \bar{P}_3^i(k,0) = \alpha\delta_{k,0} \\ \bar{P}_3^o(k,0) = (1-\alpha)\delta_{k,0}. \end{cases} \tag{A2}$$

One then straightforwardly obtains the following algebraic linear system:

$$\begin{cases} Q^i(x,s)\left(1-\frac{s}{4}(\mathrm{e}^{-\mathrm{i}x}+2\,\mathrm{e}^{\mathrm{i}x})\right)-\frac{s}{4}\,\mathrm{e}^{\mathrm{i}x}\,Q^o(x,s)=\alpha \\ -\frac{s}{4}\,\mathrm{e}^{-\mathrm{i}x}\,Q^i(x,s)+\left(1-\frac{s}{4}(\mathrm{e}^{-\mathrm{i}x}+2\,\mathrm{e}^{\mathrm{i}x})\right)Q^o(x,s)=1-\alpha \end{cases} \tag{A3}$$

determining $Q^{i,o}(x,s)$:

$$Q^{i,o}(x,s)=\frac{a_\alpha^{i,o}(x)+b_\alpha^{i,o}(x)s}{p(x,s)}. \tag{A4}$$

We omit the details irrelevant to the purpose of this appendix. We denote as $s_\pm(x)$ the roots of $p(x,s)$. They obey the equations

$$\begin{cases} s_+=2-\mathrm{i}x+O(x^2) \\ s_-=1-\frac{\mathrm{i}x}{4}+O(x^2) \end{cases} \tag{A5}$$

and one finally gets

$$\frac{\langle k\rangle_3}{n}=\frac{1}{4}. \tag{A6}$$

One now has to make sure that for any word w_n of n letters in the alphabet $\bar{\sigma}_i$ the following relation holds:

$$k(w_n)=L(w_n)+O(1). \tag{A7}$$

Even if figure 3 makes this statement clear, a more rigorous proof is as follows. Consider a given word w, with $k(w)=k_0$. Then the following decomposition holds:

$$w=b_3^{\epsilon_0}\left(\prod_{i=1}^{k_0}a_2b_3^{\epsilon_i}\right)a_2^{\epsilon_f} \tag{A8}$$

with $\epsilon_0\in\{0,1,2\}$, $\epsilon_i\in\{1,2\}$, $\epsilon_f\in\{0,1\}$. To prove (A7) we use the relation

$$L\left(\prod_{i=1}^{k_0}a_2b_3^{\epsilon_i}\right)=k_0 \tag{A9}$$

and one finally has

$$\lim_{n\to\infty}\frac{L(w_n)}{n}=\frac{1}{4}. \tag{A10}$$

Let us mention that a more direct derivation of this result can be brought in if one considers the $PSL(2,\mathbb{Z})$ generators. The structure of the Cayley graph of $PSL(2,\mathbb{Z})$ depends on the basis and in the framing S, T, T^{-1} it has the form of the so-called hyperbolic honeycomb lattice (see figure 7).

We denote by $P_n(\mu)$ the probability of the fact that the randomly generated n-letter word $w_{\{\tilde{\sigma}_1,\tilde{\sigma}_2\}}$ with the uniform distribution $\nu=\frac{1}{4}$ over the generators $\left\{\tilde{\sigma}_1,\tilde{\sigma}_2,\tilde{\sigma}_1^{-1},\tilde{\sigma}_2^{-1}\right\}$ can be contracted to the minimal irreducible word of length μ in terms of generators $\left\{\tilde{\sigma}_1,\tilde{\sigma}_2,\tilde{\sigma}_1^{-1},\tilde{\sigma}_2^{-1}\right\}$. The point is that μ coincides with the distance on the backbone graph of $PSL(2,\mathbb{Z})$.

In other words, the function $P_n(\mu)$ defines the probability of finding the random walk at a distance μ along the backbone graph from the origin after n steps. We distinguish 'left' and 'right' cells (with respect to the initial 'root' plaquette), labelled correspondingly by $\alpha=1,2$. In figure 7, the value of α is shown by the number in the centre of each honeycomb plaquette.

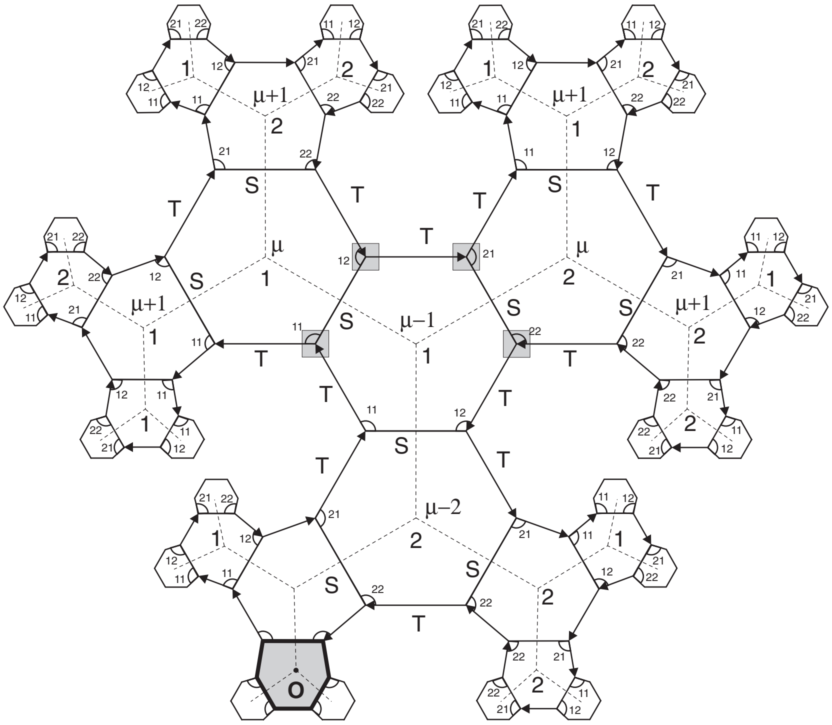

Figure 7. The honeycomb lattice.

Table 2. The elementary moves on the honeycomb graph.

$\alpha = 1$		$\alpha = 2$	
$\tilde{\sigma}_1 = T$	$\mu \to \mu + 1$	$\tilde{\sigma}_1 = T$	$\mu \to \mu - 1$
$\tilde{\sigma}_2 = T^{-1}ST^{-1}$	$\mu \to \mu$	$\tilde{\sigma}_2 = T^{-1}ST^{-1}$	$\mu \to \mu + 1$
$\tilde{\sigma}_1^{-1} = T^{-1}$	$\mu \to \mu - 1$	$\tilde{\sigma}_1^{-1} = T^{-1}$	$\mu \to \mu + 1$
$\tilde{\sigma}_2^{-1} = TS^{-1}T$	$\mu \to \mu + 1$	$\tilde{\sigma}_2^{-1} = TS^{-1}T$	$\mu \to \mu$

Suppose the walker stays in some vertex α of the cell located at a distance $\mu > 1$ from the origin along the graph backbone. The change in μ after changing one arbitrary step from the set $\left\{\tilde{\sigma}_1, \tilde{\sigma}_2, \tilde{\sigma}_1^{-1}, \tilde{\sigma}_2^{-1}\right\}$ is summarized in table 2.

It is clear that for any value of α two steps increase the length of the backbone, μ, one step decreases it and one step leaves μ without changes[5] (see table 2).

Let us introduce the effective probabilities: p_1—to jump to some specific cell among three neighbouring ones of the graph and p_2—to stay in the given cell. Because of the symmetry of the graph, the conservation law has to be written as $3p_1 + p_2 = 1$. By definition we have: $p_1 \stackrel{\text{def}}{=} \nu = \frac{1}{4}$. Thus we can write the following set of recursion relations for the integral probability $P_n(\mu, N)$:

$$P_{n+1}(\mu) = \tfrac{1}{4}P_n(\mu + 1) + \tfrac{1}{4}P_n(\mu) + \tfrac{1}{2}P_n(\mu - 1) \qquad \text{(A11)}$$

[5] In fact, each cell has two output vertices labelled by indices '11' and '12' for 'left' cells and '21' and '22' for the 'right', but as one can check straightforwardly it is not necessary to discriminate between them.

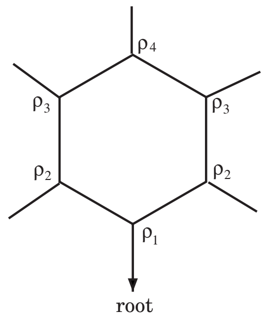

Figure 8. Different types of vertices and their corresponding weights (here $q = 6$). ρ_i gives the asymptotic ($k, n \to \infty$) probability of being at a vertex of type i.

with the following boundary conditions:

$$P_{n+1}(0) = \frac{1}{2}(P_n(0) + P_n(1)). \tag{A12}$$

This is a standard problem whose solution is known, and the condition $L(w_n) = 0$ is in particular equivalent to $\mu = 0$, therefore the probability of obtaining a trivial word after n random steps (denoted $p_r^\pi(n)$) is given by

$$p_r^\pi(n) = P_n(0) = C\frac{\lambda^n}{n^{3/2}} \tag{A13}$$

with

$$C = \frac{9+4\sqrt{2}}{7\pi} \qquad \text{and} \qquad \lambda = \frac{2\sqrt{2}+1}{4}.$$

Appendix B. General formalism for random walks on H_q

Let us generalize the computations of section 3 to the case of H_q. One can write

$$\bar{P}_q(k, n) = \sum_{i=1}^{[\frac{q}{2}]+1} \bar{P}_q^i(k, n) \tag{B1}$$

and define the constants ρ_i, $1 \leqslant i \leqslant \left[\frac{q}{2}\right] + 1$ (assuming the existence of the corresponding limits):

$$\rho_i = \lim_{k\to\infty}\left(\lim_{n\to\infty}\frac{\bar{P}_q^i(k, n)}{\bar{P}_q(k, n)}\right) \tag{B2}$$

which satisfy the normalization condition

$$\sum_{i=1}^{[\frac{q}{2}]+1} \rho_i = 1. \tag{B3}$$

The sum (B1) runs over $\left[\frac{q}{2}\right]+1$ non-equivalent vertices (the graph is locally $\mathbb{Z}_2$ symmetric, see figure 8) of the elementary q-gon of the graph. Proceeding in the standard way, we define the transform

$$Q_q^i(x, s) = \mathcal{T}\left[\bar{P}_q^i\right] \tag{B4}$$

and derive the master equation, whose solution can be expressed in the following form:

$$\bar{P}_q^i = \mathcal{T}^{-1}\left[\sum_{j=1}^{n_q} \alpha_j \left(M_q^{-1}\right)^{ji}(x, s, \vec{\rho})\right] \tag{B5}$$

where the α_j parametrize the initial conditions

$$\sum_{i=1}^{[\frac{q}{2}]+1} Q_q^i M_q^{ij}(x, s, \vec{\rho}) = \alpha_j \tag{B6}$$

and

$$M_q(x, s, \vec{\rho}) = \begin{pmatrix} -1 & \frac{s}{3}(1+\mathrm{e}^{\mathrm{i}x}) & \frac{s}{3}\,\mathrm{e}^{\mathrm{i}x} & \cdots & \cdots & \frac{s}{3}\,\mathrm{e}^{\mathrm{i}x} \\ \frac{s}{3}\left(2+\frac{\rho_2}{1-\rho_1}\,\mathrm{e}^{-\mathrm{i}x}\right) & -1 & \frac{s}{3} & 0 & \cdots & 0 \\ \frac{s}{3}\frac{\rho_3}{1-\rho_1}\,\mathrm{e}^{-\mathrm{i}x} & \frac{s}{3} & -1 & \frac{s}{3} & \ddots & \vdots \\ \frac{s}{3}\frac{\rho_4}{1-\rho_1}\,\mathrm{e}^{-\mathrm{i}x} & 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & \frac{s}{3} & -1 & \frac{s}{3} \\ \frac{s}{3}\frac{\rho_n}{1-\rho_1}\,\mathrm{e}^{-\mathrm{i}x} & 0 & \cdots & 0 & \frac{s}{3} & -1+\frac{s}{3} \end{pmatrix}. \tag{B7}$$

For $n, k \gg 1$ one obtains, using the same method as for $q = 3$

$$\bar{P}_q^i(k, n) = \bar{\rho}_i(\vec{\rho})\delta(k - \bar{l}_q(\vec{\rho})n) \tag{B8}$$

where

$$\bar{l}_q(\vec{\rho}) = \lim_{n\to\infty} \frac{\langle k\rangle_q}{n} = \mathrm{i}\frac{\mathrm{d}s_-^q}{\mathrm{d}x} \tag{B9}$$

and s_-^q is the root of the polynomial $\det(M_q(x, s, \vec{\rho}))$ closest to zero.

To make the system of equations (B5)–(B7) self-consistent we must set

$$\rho_i = \bar{\rho}_i(\vec{\rho}) \tag{B10}$$

which closes a system of equations determining ρ_i. Finally, one can write the limiting drift in the following form:

$$l_q = \lim_{n\to\infty} \frac{\langle d\rangle_q}{n} = \bar{l}_q(\vec{\rho})\left(1 + \frac{\sum_{i=2}^{n_q}(i-1)\rho_i}{1-\rho_1}\right). \tag{B11}$$

One can check that this formalism gives for $q = 3$ the same results as has been derived in section 3.

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